

On the controllability of quantum transport in an electronic nanostructure^{*}

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Abstract

We investigate the controllability of quantum electrons trapped in a two-dimensional device, typically a MOS field-effect transistor. The problem is modeled by the Schrödinger equation in a bounded domain coupled to the Poisson equation for the electrical potential. The controller acts on the system through the boundary condition on the potential, on a part of the boundary modeling the gate. We prove that, generically with respect to the shape of the domain and boundary conditions on the gate, the device is controllable. We also consider control properties of a more realistic nonlinear version of the device, taking into account the self-consistent electrostatic Poisson potential.

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1 Introduction and main results

In order to comply with the growing needs of ultra-fast, low-consumption and high-functionality operation, microelectronics industry has driven transistor sizes to the nanometer scale [4, 23, 47]. This has led to the possibility of building nanostructures like single electron transistors or single electron memories, which involve the transport of only a few electrons. In general, such devices consist in an active region (called the channel or the island) connecting two electrodes, known as the source and the drain, while the electrical potential in this active region can be tuned by a third electrode, the gate.

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In many applications, the performance of the device will depend on the possibility of controlling the electrons by acting on the gate voltage.

At the nanometer scale, quantum effects such as interferences or tunneling become important and a quantum transport model is necessary. In this paper, we analyze the controllability of a simplified mathematical model of the quantum transport of electrons trapped in a two-dimensional device, typically a MOS field-effect transistor. The problem is modeled by a single Schrödinger equation, in a bounded domain Ω with homogeneous Dirichlet boundary conditions, coupled to the Poisson equation for the electrical potential. This work is a first step towards more realistic models. For instance, throughout the paper, the self-consistent potential modeling interactions between electrons is either neglected, or (in the last section of the paper) considered as a small perturbation of the applied potential.

The control on this system is done through the boundary condition on the potential, on a part of the boundary modeling the gate. Degrees of freedom of the problem are the shape of the nanometric device and the position of the gate and its associated Dirichlet boundary conditions, modeling possible inhomogeneities: we prove that, generically with respect to these degrees of freedom, the device is controllable. We recall that Genericity is a measure of how frequently and robustly a property holds with respect to some parameters.

Controllability of general control-affine systems driven by the Schrödinger equation has been widely studied in the recent years. The first positive controllability results for infinite-dimensional quantum systems have been established by local inversion theorems and the so-called *return method* [5, 6] (see also [8] for more recent results in this direction). Other results have been obtained by Lyapunov-function techniques and combinations with local inversion results [9, 30, 37, 38, 39, 41] and by geometric control methods, using Galerkin or adiabatic approximations [10, 11, 12, 14, 19, 20]. Finally, let us conclude this necessarily incomplete list by mentioning that specific arguments have been developed to tackle physically relevant particular cases [7, 22, 33]. Let us also recall that genericity of sufficient conditions for the controllability of the Schrödinger equation has been studied in [35, 38, 40, 43].

Our analysis is based on the sufficient condition for approximate controllability obtained in [10], which requires a non-resonance condition on the spectrum of the internal Hamiltonian and a coupling property (the *connectedness chain* property) on the external control field. Genericity is proved by global perturbations, exploiting the analytic dependence of the eigenpairs of the Schrödinger operator.

1.1 The quantum transport model

The unperturbed device

Let us write a first model. In the following, Ω denotes a rectangle in the plane.

We assume without loss of generality that $\Omega = (0, \pi) \times (0, L)$ for some $L > 0$, so that, with the notations of Figure 1, one has $\Gamma_D^s = \{0\} \times [0, L]$, $\Gamma_D^d = \{\pi\} \times [0, L]$, $\Gamma_N = [0, \pi] \times \{0\}$ and $\Gamma_D^g = [0, \pi] \times \{L\}$. We set $\Gamma_D = \Gamma_D^s \cup \Gamma_D^d \cup \Gamma_D^g$. In the whole paper, the notation $\frac{\partial}{\partial \nu}$ denotes the outward normal derivative.

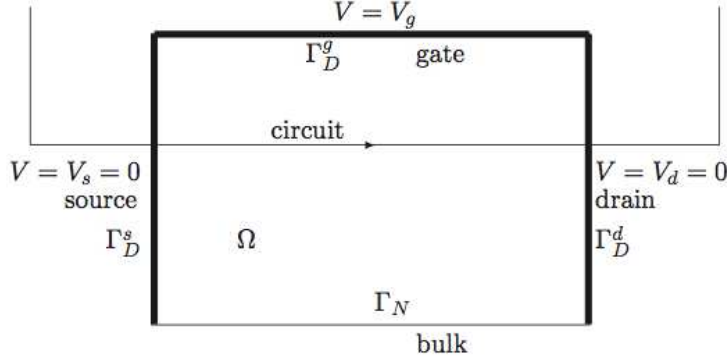


Figure 1: Representation of the transistor

We focus on the control problem

$$\left\{ \begin{array}{ll} i\partial_t \psi(t, x) = -\Delta \psi(t, x) + V(t, x)\psi(t, x), & (t, x) \in \mathbb{R}_+ \times \Omega \\ -\Delta V(t, x) = 0, & (t, x) \in \mathbb{R}_+ \times \Omega, \\ \psi(t, x) = 0, & (t, x) \in \mathbb{R}_+ \times \partial\Omega, \\ V(t, x) = \chi(x)V_g(t), & (t, x) \in \mathbb{R}_+ \times \Gamma_D^g, \\ V(t, x) = V_s = 0, & (t, x) \in \mathbb{R}_+ \times \Gamma_D^s, \\ V(t, x) = V_d = 0, & (t, x) \in \mathbb{R}_+ \times \Gamma_D^d, \\ \frac{\partial V}{\partial \nu}(t, x) = 0, & (t, x) \in \mathbb{R}_+ \times \Gamma_N. \end{array} \right. \quad (1.1)$$

The factor χ is an approximation of the constant function $\mathbb{1}_{Q(\Gamma_D^g)}$ that models spatial inhomogeneities, and is assumed to belong to

$$\mathcal{C}_0^1(\Gamma_D^g) = \{\chi \in \mathcal{C}^1(\Gamma_D^g) : \chi(0, L) = \chi(\pi, L) = 0\}.$$

The vanishing condition on the boundary of the gate guarantees the continuity of the Dirichlet condition in the equation for V . This, in turns, ensures that the trace of $V(t, \cdot)$ on $\Gamma_D^s \cup \Gamma_D^g \cup \Gamma_D^d$ belongs to $H^{1/2}(\Gamma_D^s \cup \Gamma_D^g \cup \Gamma_D^d)$, and then $V(t, \cdot) \in H^1(\Omega)$.

Here, ψ is the wave function of the electrons, satisfying the Schrödinger equation with the potential V . This potential solves the Poisson equation with a vanishing right-hand side, which means that we neglect the self-consistent electrostatic effects. In Section 3.2, as a generalization, we incorporate the self-consistent potential in the model as a perturbation of the applied potential V .

Let us comment on the boundary conditions. The wavefunction ψ is subject to homogeneous Dirichlet boundary conditions, modelling the fact that the electrons are trapped in the device. For the potential V , the only nontrivial boundary condition is taken at the upper side of the rectangle Γ_D^g where the gate is located. The applied grid voltage $t \mapsto V_g(t)$, with values in $[0, \delta]$ for some $\delta > 0$ fixed throughout the paper, is seen as

a *control*, in the sense that the evolution of the system can be driven by its choice. At the source and drain contacts Γ_D^s and Γ_D^d , we impose homogeneous Dirichlet boundary conditions: we assume indeed for simplicity that $V_s = V_d = 0$, the goal of this paper being to study the possibility of controlling by the gate. Finally, a Neumann boundary condition is imposed at the lower side Γ_N of the rectangle, assumed in contact with the bulk where electrical neutrality holds.

The existence and uniqueness of mild solutions of (1.1) in $\mathcal{C}^0(\mathbb{R}, L^2(\Omega, \mathbb{C}))$ for V_g in $L^\infty(\mathbb{R}, [0, \delta])$ is then a consequence of general results for semilinear equations (see, for instance [3] or [42]).

Problem with shape inhomogeneities

The problem above can be seen as an idealization, in the sense that the shape of the device is assumed to be perfectly rectangular.

Irregularities and inhomogeneities can be introduced in the model as follows. Let Q be in

$$\text{Diff}_0^1 = \{Q : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \mid Q \text{ orientation-preserving } \mathcal{C}^1\text{-diffeomorphism}\}$$

and χ be a function in

$$\mathcal{C}_0^1(Q(\Gamma_D^g)) = \{\chi \in \mathcal{C}^1(Q(\Gamma_D^g)) : \chi(Q(0, L)) = \chi(Q(\pi, L)) = 0\}. \quad (1.2)$$

Replacing Ω by $Q(\Omega)$, the resulting system writes

$$\left\{ \begin{array}{ll} i\partial_t \psi(t, x) = -\Delta \psi(t, x) + V(t, x)\psi(t, x), & (t, x) \in \mathbb{R}_+ \times Q(\Omega), \\ -\Delta V(t, x) = 0, & (t, x) \in \mathbb{R}_+ \times Q(\Omega), \\ \psi(t, x) = 0, & (t, x) \in \mathbb{R}_+ \times Q(\partial\Omega), \\ V(t, x) = V_g(t)\chi(x), & (t, x) \in \mathbb{R}_+ \times Q(\Gamma_D^g), \\ V(t, x) = 0, & (t, x) \in \mathbb{R}_+ \times Q(\Gamma_D^s \cup \Gamma_D^d), \\ \frac{\partial V}{\partial \nu}(t, x) = 0, & (t, x) \in \mathbb{R}_+ \times Q(\Gamma_N). \end{array} \right. \quad (1.3)$$

We clearly have $V(t, x) = V_g(t)V_0^{Q, \chi}(x)$ where $V_0^{Q, \chi}$ solves

$$\left\{ \begin{array}{ll} -\Delta V_0^{Q, \chi}(x) = 0, & x \in Q(\Omega) \\ V_0^{Q, \chi}(x) = \chi, & x \in Q(\Gamma_D^g) \\ V_0^{Q, \chi}(x) = 0, & x \in Q(\Gamma_D^s \cup \Gamma_D^d) \\ \frac{\partial V_0^{Q, \chi}}{\partial \nu}(x) = 0, & x \in Q(\Gamma_N). \end{array} \right. \quad (1.4)$$

As for the unperturbed system, mild solutions of (1.3) in $\mathcal{C}^0(\mathbb{R}, L^2(Q(\Omega, \mathbb{C})))$ exist and are unique for V_g in $L^\infty(\mathbb{R}, [0, \delta])$.

1.2 Action of the grid voltage on the system

A control approach

Our aim is to understand to what extent the system can be manipulated through the grid voltage. In this perspective, the time-varying parameter $V_g(\cdot)$ is seen as a control law and the objective is to characterize the controllability properties of the resulting system.

Definition 1.1. *We say that the control system (1.3) is approximately controllable if, for every $\psi_0, \psi_1 \in L^2(Q(\Omega), \mathbb{C})$ with unit norm and every $\varepsilon > 0$, there exist a positive time T and a control $V_g \in L^\infty([0, T], [0, \delta])$ such that the solution ψ of (1.3) with initial condition $\psi(0) = \psi_0$ satisfies $\|\psi(T) - \psi_1\|_{L^2(Q(\Omega))} < \varepsilon$.*

Notice that, for quantum control systems with bounded control operators, exact controllability¹ cannot be expected (see [3, 46]). This justifies our choice of approximate controllability as a notion of arbitrary maneuverability of the system. Other possible notions of controllability considered in the literature are exact controllability between smooth enough wavefunctions (see [6, 8]) or exact controllability in infinite time (see [41]).

The issue of determining whether (1.3) is approximately controllable for a given pair (Q, χ) seems a difficult task in general, since the known sufficient criteria for approximate controllability require a fine knowledge of the spectral properties of the operators involved (see Section 2.1). Instead, our main goal is to study the controllability properties of the model which hold true *generically* with respect to the diffeomorphism Q and the boundary condition χ . Genericity is a measure of how often and with which degree of robustness a property holds. More precisely, a property described by a boolean function $P : X \rightarrow \{0, 1\}$ is said to be *generic* in a Baire space X if there exists a residual set² $Y \subset X$ such that every x in Y satisfies the property P , that is, $P(x) = 1$. Recall that a residual set is in particular dense in X .

Genericity results with respect to χ and (Q, χ)

We are now ready to state our two main results. First consider the problems of the form (1.3) where $Q = \text{Id}$, for which the genericity of the controllability is considered only with respect to variations of the boundary condition χ on the grid Γ_D^g . We allow χ to vary within the class $\mathcal{C}_0^1(\Gamma_D^g)$ defined in (1.2), whose metric is complete, making it a Baire space.

We have the following genericity result.

Theorem 1.2. *Let $L^2 \notin \pi^2\mathbb{Q}$. For $Q = \text{Id}$ and a generic χ in $\mathcal{C}_0^1(\Gamma_D^g)$, the control problem (1.3) is approximately controllable.*

Consider now the entire class of problems of the form (1.3). In order to endow it with a topological structure, we identify (1.3) with the triple $(Q(\Omega), Q(\Gamma_D^g), \chi)$. The family of

¹System (1.3) would be *exactly controllable* if for every $\psi_0, \psi_1 \in L^2(Q(\Omega), \mathbb{C})$ with unit norm, there existed a positive time T and a control $V_g \in L^\infty([0, T], [0, \delta])$ such that $\psi(T) = \psi_1$, where ψ denotes the solution of (1.3) corresponding to V_g with initial condition $\psi(0) = \psi_0$.

²i.e. the intersection of countably many open and dense subsets.

problems is then given by

$$\mathcal{P} = \{(Q(\Omega), Q(\Gamma_D^g), \chi) \mid (Q(\Omega), Q(\Gamma_D^g)) \in \Sigma \text{ and } \chi \in \mathcal{C}_0^1(Q(\Gamma_D^g))\}, \quad (1.5)$$

where

$$\Sigma = \{(Q(\Omega), Q(\Gamma_D^g)) \mid Q \in \text{Diff}_0^1\}.$$

The metric induced³ by that of \mathcal{C}^1 -diffeomorphisms and by the \mathcal{C}^1 metric on $\mathcal{C}_0^1(\Gamma_D^g)$ makes \mathcal{P} complete ([36]). In particular, \mathcal{P} is a Baire space.

Theorem 1.3. *For a generic element of \mathcal{P} , the control problem (1.3) is approximately controllable.*

The proofs of Theorems 1.2 and 1.3 can be found in Sections 2.4 and 2.5, respectively. They are based on a general sufficient condition for controllability proved in [10] and recalled in Section 2.1 below. In a nutshell, such a condition is based, on the one hand, on a nonresonance property of the spectrum of the Schrödinger operator and, on the other hand, on a coupling property for the interaction term (see the notion of connectedness chain introduced in Definition 2.1). These properties are expressed as a countable number of open conditions. Their density is proved through a global analytic propagation argument.

In Section 3, we present two generalizations of these results, motivated by the applications. First, in Subsection 3.1, we consider a situation where the gate only partially covers the upper side of the rectangle domain. Then, in Subsection 3.2, we take into account in our model the self-consistent electrostatic Poisson potential, as a perturbation of the applied potential V .

2 Proof of the genericity results

2.1 General controllability conditions for bilinear quantum systems

We recall in this section a general approximate controllability result for bilinear quantum systems obtained in [10].

Let \mathcal{H} be a complex Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and A, B be two linear skew-adjoint operators on \mathcal{H} . Let B be bounded and denote by $D(A)$ the domain of A . Consider the controlled equation

$$\frac{d\psi}{dt}(t) = (A + u(t)B)\psi(t), \quad u(t) \in [0, \delta], \quad (2.1)$$

with $\delta > 0$. We say that A satisfies assumption (A) if there exists an orthonormal basis $(\phi_k)_{k \in \mathbb{N}}$ of \mathcal{H} made of eigenvectors of A whose associated eigenvalues $(i\lambda_k)_{k \in \mathbb{N}}$ are all simple.

³This metric is defined, for $(\Omega_1, \Gamma_1, \chi_1)$ and $(\Omega_2, \Gamma_2, \chi_2)$ in \mathcal{P} , by

$$\inf \left\{ \|Q_1 - Q_2\|_{\mathcal{C}^1(\Omega, \mathbb{R}^2)} + \|\chi_1 \circ Q_1 - \chi_2 \circ Q_2\|_{\mathcal{C}^1(\Gamma_D^g, \mathbb{R})} \mid Q_j \in \text{Diff}_0^1, Q_j(\Omega) = \Omega_j, Q_j(\Gamma_D^g) = \Gamma_j, j = 1, 2 \right\}.$$

Definition 2.1. A subset S of \mathbb{N}^2 couples two levels j, k in \mathbb{N} if there exists a finite sequence $((s_1^1, s_2^1), \dots, (s_1^p, s_2^p))$ in S such that

- (i) $s_1^1 = j$ and $s_2^p = k$;
- (ii) $s_2^j = s_1^{j+1}$ for every $1 \leq j \leq p-1$.

S is called a *connectedness chain* if S couples every pair of levels in \mathbb{N} .

S is a *non-resonant connectedness chain* for (A, B, Φ) if it is a *connectedness chain*, $\langle \phi_j, B\phi_k \rangle \neq 0$ for every $(j, k) \in S$, and $\lambda_{s_1} - \lambda_{s_2} \neq \lambda_{t_1} - \lambda_{t_2}$ for every $(s_1, s_2) \in S$ with $s_1 \neq s_2$ and every (t_1, t_2) in $\mathbb{N}^2 \setminus \{(s_1, s_2)\}$ such that $\langle \phi_{t_1}, B\phi_{t_2} \rangle \neq 0$.

Theorem 2.2 ([10]). Let A satisfy (\mathfrak{A}) and let $\Phi = (\phi_k)_{k \in \mathbb{N}}$ be an orthonormal basis of eigenvectors of A . If there exists a non-resonant connectedness chain for (A, B, Φ) then (2.1) is approximately controllable.

Remark 1. The simplicity of the spectrum required in Definition 2.1 is not necessary. The construction in [10] is indeed slightly more general and we refer to that paper and [11] for further details.

We also recall that a similar result based on a stronger requirement has been proposed in [20]. In that paper, the spectrum of the operator A was asked to be *non-resonant*, in the sense that every nontrivial finite linear combination with rational coefficients of its eigenvalues was asked to be nonzero.

Remark 2. The statement of Theorem 2.2 could be strengthened, according to the results in [10], in two other directions: first, the controllability could be extended beyond single wavefunctions, towards ensembles (controllability in the sense of density matrices and simultaneous controllability); second, unfeasible trajectories in the unit sphere of \mathcal{H} turn out to be trackable (i.e., they can be followed approximately with arbitrary precision by admissible ones) at least when the modulus (but not the phase) of the components of the wavefunction are considered. Moreover, the proof of Theorem 2.2 given in [10] is constructive, leading to a control design algorithm based on the knowledge of the spectrum of the operator A (see also [19] for an alternative construction).

Remark 3. Another consequence of the Lie–Galerkin approach behind Theorem 2.2 is that the conclusions of Theorems 1.2 and 1.3 could be strengthened by stating approximate controllability in stronger topologies provided that $V_0^{Q, \chi}$ belongs to $H^2(Q(\Omega))$ (this is always the case in the framework of Theorem 1.2 as it follows from standard elliptic regularity results in rectangles). The key point is that approximate controllability can be obtained by requiring, in addition, that the total variation and the L^1 norm of the control law are bounded uniformly with respect to the tolerance (see [10, 19]). Proposition 3 in [13] and Proposition 6 in [14] then implies that, for an initial and final conditions $\psi_0, \psi_1 \in H^2(\Omega)$, for every tolerance $\varepsilon > 0$, there exists a control steering ψ_0 to an ε -neighbourhood of ψ_1 for the L^2 -norm, while satisfying a uniform bound (independent of ε) for the H^2 -norm. An interpolation argument allows to conclude that, for $\xi \in (0, 2)$, ψ_0 can be steered ε -close to ψ_1 in the H^ξ -norm.

2.2 Preliminary steps of the proofs

The proofs of Theorems 1.2 and 1.3 are based on the idea of *propagating* sufficient controllability conditions using analytic perturbations ([28, 35, 43]). This is possible since the general controllability criterion for quantum systems seen in the previous section can be seen as a countable set of nonvanishing scalar conditions.

More precisely, let us denote by $\Lambda(Q(\Omega))$ the spectrum of the Laplace–Dirichlet operator on $Q(\Omega)$ and, for every $(Q(\Omega), Q(\Gamma_D^g), \chi) \in \mathcal{P}$ such that $\Lambda(Q(\Omega))$ is simple (i.e., each eigenvalue is simple), define

$$S(Q(\Omega), Q(\Gamma_D^g), \chi) = \left\{ (k, j) \in \mathbb{N}^2 \mid \int_{Q(\Omega)} V_0^{Q, \chi}(x) \phi_k(x) \phi_j(x) dx \neq 0 \right\}, \quad (2.2)$$

where $\{\phi_j\}_{j \in \mathbb{N}}$ is a Hilbert basis of eigenfunctions of the Laplace–Dirichlet operator on $Q(\Omega)$, ordered following the growth of the corresponding eigenvalues.

Theorem 1.3 is proved by applying Theorem 2.2 with A the Laplace–Dirichlet operator on $Q(\Omega)$ multiplied by i and B the multiplicative operator defined by $B\psi = -iV_0^{Q, \chi}\psi$. We then show that both sets

$$\mathcal{P}_1 = \{(Q(\Omega), Q(\Gamma_D^g), \chi) \in \mathcal{P} \mid \Lambda(Q(\Omega)) \text{ non-resonant}\},$$

where the notion of non-resonant spectrum is the one introduced in Remark 1, and

$$\mathcal{P}_2 = \{(Q(\Omega), Q(\Gamma_D^g), \chi) \in \mathcal{P} \mid \Lambda(Q(\Omega)) \text{ simple, } S(Q(\Omega), Q(\Gamma_D^g), \chi) \text{ connectedness chain}\}$$

are residual in \mathcal{P} . Their intersection is therefore residual as well (it is itself the intersection of countably many open dense sets). The following result resumes these considerations.

Proposition 2.3. *If \mathcal{P}_1 and \mathcal{P}_2 are residual then the control problem (1.3) is approximately controllable for a generic element of \mathcal{P} .*

The situation is slightly different for the proof of Theorem 1.2, since the fact that $Q = \text{Id}$ prevents $\Lambda(Q(\Omega)) = \Lambda(\Omega)$ from being non-resonant. Recall that $(0, \delta)$ is the interval of admissible control values (see Section 1.1). We are then led to rewrite, for every $\rho \in [0, \delta)$, equation (1.3) in the case $Q = \text{Id}$ as

$$\begin{cases} i\partial_t \psi(t, x) = (-\Delta + \rho V_0^{\text{Id}, \chi}(x))\psi(t, x) + (V_g(t) - \rho)V_0^{\text{Id}, \chi}(x)\psi(t, x), & (t, x) \in \mathbb{R}_+ \times \Omega, \\ \psi(t, x) = 0, & (t, x) \in \mathbb{R}_+ \times \partial\Omega. \end{cases} \quad (2.3)$$

We apply Theorem 2.2 to (2.3) with $A = -i(-\Delta + \rho V_0^{\text{Id}, \chi}\text{Id})$ on Ω (with Dirichlet boundary conditions) and $B = -iV_0^{\text{Id}, \chi}\text{Id}$. In analogy to the notation introduced above, let

$$\mathcal{P}_{1, \text{BC}}^\rho = \{\chi \in \mathcal{C}_0^1(\Gamma_D^g) \mid \text{the spectrum of } -\Delta + \rho V_0^{\text{Id}, \chi}\text{Id} \text{ is weakly non-resonant}\},$$

where a sequence $(\lambda_n)_{n \in \mathbb{N}}$ is said to be *weakly non-resonant* if $\lambda_{s_1} - \lambda_{s_2} \neq \lambda_{t_1} - \lambda_{t_2}$ for every $(s_1, s_2), (t_1, t_2) \in \mathbb{N}^2$ with $s_1 \neq s_2$ and $(s_1, s_2) \neq (t_1, t_2)$.

Moreover, let

$$\mathcal{P}_{2,\text{BC}}^\rho = \{\chi \in \mathcal{C}_0^1(\Gamma_D^g) \mid \text{the spectrum of } -\Delta + \rho V_0^{\text{Id},\chi} \text{Id is simple} \\ \text{and } S_\rho(\chi) \text{ is a connectedness chain}\}$$

where

$$S_\rho(\chi) = \left\{ (k, j) \in \mathbb{N}^2 \mid \int_{\Omega} V_0^{\text{Id},\chi}(x) \phi_{k,\rho}(x) \phi_{j,\rho}(x) dx \neq 0 \right\} \quad (2.4)$$

and $\{\phi_{j,\rho}\}_{j \in \mathbb{N}}$ is a Hilbert basis of eigenfunctions of $-\Delta + \rho V_0^{\text{Id},\chi} \text{Id}$, ordered following the growth of the corresponding eigenvalues.

System (2.3) is approximately controllable if $\chi \in \mathcal{P}_{1,\text{BC}}^\rho \cap \mathcal{P}_{2,\text{BC}}^\rho$ for some $\rho \in (0, \delta)$.

Theorem 1.2 is then proved through the following proposition, playing the role of Proposition 2.3 in the case $Q = \text{Id}$.

Proposition 2.4. *Let $L^2 \notin \pi^2 \mathbb{Q}$ and $Q = \text{Id}$. If there exists $\rho \in (0, \delta)$ such that $\mathcal{P}_{1,\text{BC}}^\rho$ and $\mathcal{P}_{2,\text{BC}}^\rho$ are residual then the control problem (1.3) is approximately controllable for a generic χ in $\mathcal{C}_0^1(\Gamma_D^g)$.*

A crucial tool for proving that the sets introduced above are residual is the following proposition, stating that $V_0^{Q,\chi}$ is analytic with respect to Q and χ .

Proposition 2.5. *Let I be an open interval and $I \ni t \mapsto (Q_t, \varphi_t)$ be an analytic curve in the product of Diff_0^1 with the space $\mathcal{C}_0^1(\Gamma_D^g)$ defined in (1.2). Denote by χ_t the composition $\varphi_t \circ Q_t^{-1}$ and by $V_{0,t}$ the function $V_0^{Q_t, \chi_t}$ defined as in (1.4). Then $t \mapsto V_{0,t} \circ Q_t$ is an analytic curve in $H^1(\Omega)$.*

The proof of the proposition is given in next section. One important consequence for our argument is the following corollary.

Corollary 2.6. *Let $\hat{\mathcal{P}}$ be one of the sets $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_{1,\text{BC}}^\rho, \mathcal{P}_{2,\text{BC}}^\rho$. If $\hat{\mathcal{P}}$ is nonempty, then $\hat{\mathcal{P}}$ is residual. Moreover, if $\mathcal{P}_{j,\text{BC}}^\rho$ is nonempty for $j \in \{1, 2\}$ and $\rho \in [0, \delta)$, then $\mathcal{P}_{j,\text{BC}}^{\rho'}$ is nonempty (and hence dense) for almost all $\rho' \in [0, \delta)$.*

Proof of Corollary 2.6. In order to avoid redundancies, we prove the corollary only in the case $\hat{\mathcal{P}} = \mathcal{P}_2$. The proof can be easily adapted to the other cases.

Let us first prove that \mathcal{P}_2 is the intersection of countably many open sets. We claim that $\mathcal{P}_2 = \bigcap_{n \in \mathbb{N}} \mathcal{A}_n$, where \mathcal{A}_n is the set of triples $(Q(\Omega), Q(\Gamma_D^g), \chi) \in \mathcal{P}$ such that the first n eigenvalues of the Laplace–Dirichlet operator on $Q(\Omega)$ are simple and there exist $r \in \mathbb{N}$ and r other simple eigenvalues of $\lambda_{k_1}, \dots, \lambda_{k_r}$ such that the matrix

$$\left(\int_{Q(\Omega)} \phi_j \phi_l V_0^{Q,\chi} \right)_{j,l \in \{1, \dots, n\} \cup \{k_1, \dots, k_r\}} \quad (2.5)$$

is connected⁴, where each ϕ_j is an eigenfunction corresponding to λ_j . It is clear that an element of $\bigcap_{n \in \mathbb{N}} \mathcal{A}_n$ is in \mathcal{P}_2 , since its corresponding spectrum is simple and a connectedness

⁴We recall that a $m \times m$ matrix $C = (c_{jl})_{j,l=1}^m$ is said to be *connected* if for every pair of indices $j, l = 1, \dots, m$ there there exists a finite sequence $j_1, \dots, j_w \in \{1, \dots, m\}$ such that $c_{jj_1} c_{j_1 j_2} \cdots c_{j_{w-1} j_w} c_{j_w l} \neq 0$. The set $\{(j, l) \mid c_{jl} \neq 0\}$ is said to be a *connectedness chain* for C .

chain is given by the union of all the connectedness chains for the matrices of the type (2.5). Conversely, if $(Q(\Omega), Q(\Gamma_D^g), \chi) \in \mathcal{P}_2$, then there exists a bijection $\xi : \mathbb{N} \rightarrow \mathbb{N}$ such that each matrix

$$\left(\int_{Q(\Omega)} \phi_{\xi(j)} \phi_{\xi(l)} V_0^{Q, \chi} \right)_{j, l \in \{1, \dots, n\}}$$

is connected (see [35, Remark 4.2]). Given $n \in \mathbb{N}$, let N be such that $\xi(\{1, \dots, N\}) \supset \{1, \dots, n\}$. Then, taking $r = N - n$ and $\{\lambda_{k_1}, \dots, \lambda_{k_r}\} = \{\lambda_{\xi(1)}, \dots, \lambda_{\xi(N)}\} \setminus \{\lambda_1, \dots, \lambda_n\}$, we have that $(Q(\Omega), Q(\Gamma_D^g), \chi) \in \mathcal{A}_n$.

Since each \mathcal{A}_n is open (by continuity of the eigenpairs corresponding to simple eigenvalues), we have proved that \mathcal{P}_2 is the intersection of countably many open sets.

Let us now show that \mathcal{P}_2 is dense if it is nonempty. Fix $(Q(\Omega), Q(\Gamma_D^g), \chi) \in \mathcal{P}_2$ and let

$$\bar{S} = S(Q(\Omega), Q(\Gamma_D^g), \chi).$$

Let $I \ni t \mapsto (Q_t, \varphi_t)$ be an analytic curve in the product $\text{Diff}_0^1 \times \mathcal{C}_0^1(\Gamma_D^g)$ and assume that there exists $t_0 \in I$ such that $Q_{t_0} = Q$ and $\varphi_{t_0} \circ Q = \chi$. According to Rellich's theorem (see [32, 45]), there exists $I \ni t \mapsto (\lambda_j(t), \phi_j(t))_{j \in \mathbb{N}}$ such that $(\lambda_j(t), \phi_j(t))_{j \in \mathbb{N}}$ is a complete family of eigenpairs of the Laplace–Dirichlet operator on $Q_t(\Omega)$ for every $t \in I$, with $I \ni t \mapsto \lambda_j(t)$ and $I \ni t \mapsto \phi_j(t) \circ Q_t$ analytic in \mathbb{R} and in $L^2(\Omega, \mathbb{R})$, respectively, for every $j \in \mathbb{N}$.

Proposition 2.5 implies that for every $j, k \in \mathbb{N}$, the function

$$t \mapsto \int_{Q_t(\Omega)} \phi_j(t) \phi_k(t) V_0^{Q_t, \varphi_t \circ Q_t^{-1}}$$

is analytic on I . Moreover, the spectrum $\Lambda(Q_t(\Omega))$ is simple for almost every $t \in I$.

We can assume that the sequence $(\lambda_j(t_0))_{j \in \mathbb{N}}$ is (strictly) increasing. For every $t \in I$ such that $\Lambda(Q_t(\Omega))$ is simple, there exists $\xi_t : \mathbb{N} \rightarrow \mathbb{N}$ bijective such that $(\lambda_{\xi_t(j)}(t))_{j \in \mathbb{N}}$ is increasing. By analyticity of $t \mapsto \int_{Q_t(\Omega)} \phi_j(t) \phi_k(t) V_0^{Q_t, \varphi_t \circ Q_t^{-1}}$ for each $(j, k) \in \bar{S}$, we have that

$$\{(\xi_t^{-1}(j), \xi_t^{-1}(k)) \mid (j, k) \in \bar{S}\} \subset S(Q_t(\Omega), Q_t(\Gamma_D^g), \varphi_t \circ Q_t^{-1})$$

for almost every $t \in I$. Since for every bijection $\hat{\xi} : \mathbb{N} \rightarrow \mathbb{N}$ the set $\{(\hat{\xi}(j), \hat{\xi}(k)) \mid (j, k) \in \bar{S}\}$ is a connectedness chain, we conclude that for almost every $t \in I$, $S(Q_t(\Omega), Q_t(\Gamma_D^g), \varphi_t \circ Q_t^{-1})$ is a connectedness chain. Hence, for almost every $t \in I$, $(Q_t(\Omega), Q_t(\Gamma_D^g), \varphi_t \circ Q_t^{-1}) \in \mathcal{P}_2$.

We conclude on the density of \mathcal{P}_2 by considering all analytic curves $t \mapsto (Q_t, \varphi_t)$ passing through (Q, χ) . Indeed, given an element $(Q(\Omega), Q(\Gamma_D^g), \chi)$ in \mathcal{P}_2 , the set \mathcal{Q} of diffeomorphisms that can be joined to Q by an analytic curve in Diff_0^1 contains $Q \circ \text{Diff}_0^\infty$, where Diff_0^∞ denotes the set of smooth orientation-preserving diffeomorphisms (see, e.g., [1]). In particular \mathcal{Q} is dense in Diff_0^1 . Perturbing the boundary condition χ by linear interpolation, one easily gets that the elements of \mathcal{P} which can be joined through analytic paths of the type $t \mapsto (Q_t, \varphi_t)$ to $(Q(\Omega), Q(\Gamma_D^g), \chi)$ is dense in \mathcal{P} . Since almost every element of an analytic path in \mathcal{P} through $(Q(\Omega), Q(\Gamma_D^g), \chi)$ is in \mathcal{P}_2 , then \mathcal{P}_2 is dense in \mathcal{P} .

The second part of the statement is proved by analogous analyticity considerations with respect to the parameter ρ . \square

2.3 Proof of Proposition 2.5

Denote by $\hat{\varphi}_t$ the extension of φ_t on $\bar{\Omega}$ which is constant on every vertical segment. Then $t \mapsto \hat{\varphi}_t$ is an analytic curve in $\mathcal{C}^1(\bar{\Omega})$ with $\hat{\varphi}_t \equiv 0$ on $\Gamma_D^s \cup \Gamma_D^d$.

Define $\hat{\chi}_t = \hat{\varphi}_t \circ Q_t^{-1}$ and let $\hat{V}_{0,t} = V_{0,t} - \hat{\chi}_t$. Notice that $\hat{V}_{0,t}$ is a solution to the problem

$$\begin{cases} -\Delta \hat{V}_{0,t}(x) = \Delta \hat{\chi}_t, & x \in Q_t(\Omega), \\ \hat{V}_{0,t}(x) = 0, & x \in Q_t(\Gamma_D), \\ \frac{\partial \hat{V}_{0,t}}{\partial \nu}(x) = 0, & x \in Q_t(\Gamma_N). \end{cases} \quad (2.6)$$

Equivalently,

$$\int_{Q_t(\Omega)} \nabla \hat{V}_{0,t}(x) \cdot \nabla \phi(x) dx = \int_{Q_t(\Omega)} \Delta \hat{\chi}_t(x) \phi(x) dx,$$

for every $\phi \in H_{0,Q_t(\Gamma_D)}^1(Q_t(\Omega))$, where

$$H_{0,Q_t(\Gamma_D)}^1(Q_t(\Omega)) = \{\phi \in H^1(Q_t(\Omega)) \mid \phi = 0 \text{ on } Q_t(\Gamma_D)\}.$$

Fix $t_0 \in I$ and notice that, for every $t \in I$,

$$H_{0,Q_t(\Gamma_D)}^1(Q_t(\Omega)) = \{\phi \circ Q_{t_0} \circ Q_t^{-1} \mid \phi \in H_{0,Q_{t_0}(\Gamma_D)}^1(Q_{t_0}(\Omega))\}.$$

Set $R_t = Q_t \circ Q_{t_0}^{-1}$. By the standard change of coordinates formula,

$$\int_{Q_{t_0}(\Omega)} ((DR_t^T)^{-1} \nabla W_t) \cdot ((DR_t^T)^{-1} \nabla \phi) J_t = \int_{Q_{t_0}(\Omega)} (\Delta \hat{\chi}_t \circ R_t) \phi J_t$$

for every $\phi \in H_{0,Q_{t_0}(\Gamma_D)}^1(Q_{t_0}(\Omega))$, where DR_t and DR_t^T are, respectively, the Jacobian matrix of R_t and its transpose, while $W_t = \hat{V}_{0,t} \circ R_t$ and $J_t = \det(DR_t)$.

In other words, (t, W_t) is the solution of $F(t, W_t) = 0 \in H_{0,Q_{t_0}}^{-1}(Q_{t_0}(\Omega))$, where $H_{0,Q_{t_0}}^{-1}(Q_{t_0}(\Omega))$ stands for the dual space of $H_{0,Q_{t_0}}^1(Q_{t_0}(\Omega))$ with respect to the pivot space $L^2(Q_{t_0}(\Omega))$, with

$$\begin{aligned} F(t, W) &= -\operatorname{div}(A_t \nabla W) - (\Delta \hat{\chi}_t \circ R_t) J_t, \\ A_t &= J_t (DR_t)^{-1} (DR_t^T)^{-1}. \end{aligned}$$

The analyticity of W_t with respect to t follows by the implicit function theorem, since F is analytic from $I \times H_{0,Q_{t_0}(\Gamma_D)}^1(Q_{t_0}(\Omega))$ into $H_{0,Q_{t_0}}^{-1}(Q_{t_0}(\Omega))$ and the operator $D_W F(t_0, W_{t_0})$ is an isomorphism of $H_{0,Q_{t_0}(\Gamma_D)}^1(Q_{t_0}(\Omega))$ into $H_{0,Q_{t_0}}^{-1}(Q_{t_0}(\Omega))$. Indeed, by linearity of F with respect to W and because R_{t_0} is the identity, $D_W F(t_0, W_{t_0})Z$ is nothing else than $-\Delta Z$, which is an isomorphism from $H_{0,Q_{t_0}}^1(Q_{t_0}(\Omega))$ to $H_{0,Q_{t_0}}^{-1}(Q_{t_0}(\Omega))$, by Lax-Milgram's lemma.

This concludes the proof of Proposition 2.5.

2.4 Proof of Theorem 1.2

Notice that the assumption $L^2 \notin \pi^2\mathbb{Q}$ guarantees that the spectrum of the Laplace–Dirichlet operator on Ω is simple.

According to Proposition 2.4 and Corollary 2.6, we are left to prove that there exist $\rho_1, \rho_2 \in [0, \delta)$ such that $\mathcal{P}_{1,\text{BC}}^{\rho_1}$ and $\mathcal{P}_{2,\text{BC}}^{\rho_2}$ are nonempty. The second part of the statement of Corollary 2.6, indeed, implies then that there exists $\rho \in (0, \delta)$ such that $\mathcal{P}_{1,\text{BC}}^\rho \cap \mathcal{P}_{2,\text{BC}}^\rho$ is residual.

The proof that $\mathcal{P}_{1,\text{BC}}^\rho$ is nonempty for some $\rho \in (0, \delta)$ is made in Section 2.4.1, while it is shown in Section 2.4.2 that $\mathcal{P}_{2,\text{BC}}^0$ is nonempty.

2.4.1 There exists $\rho \in (0, \delta)$ such that $\mathcal{P}_{1,\text{BC}}^\rho$ is nonempty

Let n denote a positive integer and $\chi_n \in \mathcal{C}^\infty(\Gamma_D^g)$ be defined by

$$\chi_n(x_1, L) = \cosh(nL) \sin(nx_1). \quad (2.7)$$

Notice that in this case the solution V_0^{Id, χ_n} of (1.4) is explicitly given by

$$V_0^{\text{Id}, \chi_n}(x) = \sin(nx_1) \cosh(nx_2), \quad x = (x_1, x_2) \in \Omega.$$

Proposition 2.7. *Let $L^2 \notin \pi^2\mathbb{Q}$. If n is odd, then χ_n is in $\mathcal{P}_{1,\text{BC}}^\rho$ for almost every $\rho \in (0, \delta)$.*

Proof. The eigenpairs of the Laplace–Dirichlet operator on Ω are naturally parameterized over \mathbb{N}^2 as follows: for every $\mathbf{j} = (j_1, j_2) \in \mathbb{N}^2$, let

$$\lambda_{\mathbf{j}} = j_1^2 + j_2^2 \frac{\pi^2}{L^2}, \quad \phi_{\mathbf{j}}(x) = \frac{2}{\sqrt{\pi L}} \sin(j_1 x_1) \sin\left(j_2 \frac{\pi}{L}\right).$$

For every $\rho \in [0, \delta)$, denote by $(\lambda_{\mathbf{j}}(\rho))_{\mathbf{j} \in \mathbb{N}^2}$ the spectrum of $-\Delta + \rho V_0^{\text{Id}, \chi_n} \text{Id}$. Moreover, notice that each function $\lambda_{\mathbf{j}}(\cdot)$ can be chosen to be analytic on $[0, \delta)$, with $\lambda_{\mathbf{j}}(0) = \lambda_{\mathbf{j}} = j_1^2 + j_2^2 \frac{\pi^2}{L^2}$. Indeed, local analyticity follows from Kato–Rellich theorem (see [44, Theorem XII.13]) and global one from the fact that the derivative of each $\lambda_{\mathbf{j}}(\cdot)$ (which can be computed as in (2.8) below) is uniformly bounded.

Let us evaluate the derivative of each $\lambda_{\mathbf{j}}(\rho)$ at $\rho = 0$. Denote $\alpha_{\mathbf{j}} = \left. \frac{d\lambda_{\mathbf{j}}}{d\rho} \right|_{\rho=0}$. Recall that the derivative of the eigenvalues can be computed according to the formula

$$\alpha_{\mathbf{j}} = \int_{\Omega} V_0^{\text{Id}, \chi_n}(x_1, x_2) \phi_{\mathbf{j}}(x_1, x_2)^2 dx_1 dx_2 \quad (2.8)$$

(see, for instance, [26]).

We assume that

$$\lambda_{\mathbf{j}}(0) - \lambda_{\mathbf{k}}(0) = \lambda_{\mathbf{j}'}(0) - \lambda_{\mathbf{k}'}(0), \quad (2.9)$$

$$\alpha_{\mathbf{j}} - \alpha_{\mathbf{k}} = \alpha_{\mathbf{j}'} - \alpha_{\mathbf{k}'} \quad (2.10)$$

for some $\mathbf{j}, \mathbf{k}, \mathbf{j}', \mathbf{k}' \in \mathbb{N}^2$ with $(\mathbf{k}, \mathbf{j}) \neq (\mathbf{k}', \mathbf{j}')$ and we show that $(\mathbf{k}, \mathbf{k}') = (\mathbf{j}, \mathbf{j}')$. By analyticity we then have that $\lambda_{\mathbf{j}}(\rho) - \lambda_{\mathbf{k}}(\rho) = \lambda_{\mathbf{j}'}(\rho) - \lambda_{\mathbf{k}'}(\rho)$ only for isolated values of $\rho \in (0, \delta)$ and the proposition follows by the countability of $\mathbb{N}^2 \times \mathbb{N}^2$.

According to (2.9), we have

$$\frac{j_1^2}{\pi^2} + \frac{j_2^2}{L^2} - \frac{k_1^2}{\pi^2} - \frac{k_2^2}{L^2} = \frac{j_1'^2}{\pi^2} + \frac{j_2'^2}{L^2} - \frac{k_1'^2}{\pi^2} - \frac{k_2'^2}{L^2}.$$

Since $L^2 \notin \pi^2\mathbb{Q}$, we get

$$j_1^2 - k_1^2 = j_1'^2 - k_1'^2, \quad (2.11)$$

$$j_2^2 - k_2^2 = j_2'^2 - k_2'^2. \quad (2.12)$$

Computing (2.8) using the expression

$$V_0^{\text{Id}, \chi^n}(x_1, x_2) = \sin(nx_1) \cosh(nx_2),$$

we have

$$\begin{aligned} \alpha_{\mathbf{j}} &= \frac{4}{L\pi} \int_0^\pi \sin(nx_1) \sin(j_1 x_1)^2 dx_1 \int_0^L \cosh(nx_2) \sin\left(\frac{j_2 \pi x_2}{L}\right)^2 dx_2 \\ &= -\frac{32L\pi \sinh(nL)}{n^2} \frac{j_1^2}{4j_1^2 - n^2} \frac{j_2^2}{(2\pi)^2 j_2^2 + n^2}. \end{aligned}$$

Hence, we can rewrite (2.10) as

$$\begin{aligned} &\frac{j_1^2 j_2^2}{(4j_1^2 - n^2)((2\pi)^2 j_2^2 + n^2)} - \frac{k_1^2 k_2^2}{(4k_1^2 - n^2)((2\pi)^2 k_2^2 + n^2)} \\ &= \frac{j_1'^2 j_2'^2}{(4j_1'^2 - n^2)((2\pi)^2 j_2'^2 + n^2)} - \frac{k_1'^2 k_2'^2}{(4k_1'^2 - n^2)((2\pi)^2 k_2'^2 + n^2)}, \end{aligned} \quad (2.13)$$

from which we obtain, up to reduction to common denominator,

$$\begin{aligned} &j_1^2 j_2^2 (4k_1^2 - n^2)(4j_1'^2 - n^2)(4k_1'^2 - n^2)((2\pi)^2 k_2^2 + n^2)((2\pi)^2 j_2'^2 + n^2)((2\pi)^2 k_2'^2 + n^2) \\ &- k_1^2 k_2^2 (4j_1^2 - n^2)(4j_1'^2 - n^2)(4k_1'^2 - n^2)((2\pi)^2 j_2^2 + n^2)((2\pi)^2 j_2'^2 + n^2)((2\pi)^2 k_2'^2 + n^2) \\ &- j_1'^2 j_2'^2 (4j_1^2 - n^2)(4k_1^2 - n^2)(4k_1'^2 - n^2)((2\pi)^2 j_2^2 + n^2)((2\pi)^2 k_2^2 + n^2)((2\pi)^2 k_2'^2 + n^2) \\ &+ k_1'^2 k_2'^2 (4j_1^2 - n^2)(4j_1'^2 - n^2)(4k_1^2 - n^2)((2\pi)^2 j_2^2 + n^2)((2\pi)^2 j_2'^2 + n^2)((2\pi)^2 k_2^2 + n^2) = 0. \end{aligned}$$

We can rewrite the latter expression in the form $P(2\pi) = 0$ where P is an integer polynomial of degree at most 6.

Since 2π is a transcendental number, we necessarily have $P = 0$. In particular, its leading coefficient vanishes, that is,

$$\begin{aligned} 0 &= j_2^2 k_2^2 j_2'^2 k_2'^2 (j_1^2 (4k_1^2 - n^2)(4j_1'^2 - n^2)(4k_1'^2 - n^2) - k_1^2 (4j_1^2 - n^2)(4j_1'^2 - n^2)(4k_1'^2 - n^2) \\ &\quad - j_1'^2 (4j_1^2 - n^2)(4k_1^2 - n^2)(4k_1'^2 - n^2) + k_1'^2 (4j_1^2 - n^2)(4k_1^2 - n^2)(4j_1'^2 - n^2)). \end{aligned} \quad (2.14)$$

A simple computation leads to

$$(k_1^2 - k_1'^2)(4j_1^2 - n^2)(4j_1'^2 - n^2) = (j_1^2 - j_1'^2)(4k_1^2 - n^2)(4k_1'^2 - n^2). \quad (2.15)$$

Recall that we are assuming $(\mathbf{k}, \mathbf{j}) \neq (\mathbf{k}', \mathbf{j}')$ and that we want to prove that $(\mathbf{k}, \mathbf{k}') = (\mathbf{j}, \mathbf{j}')$. Assume for now that

$$k_1 \neq k_1'. \quad (2.16)$$

According to (2.11) we also have $j_1 \neq j_1'$. Equation (2.15), moreover, yields

$$4(k_1^2 k_1'^2 - j_1^2 j_1'^2) = n^2(k_1^2 + k_1'^2 - j_1^2 - j_1'^2).$$

Using again (2.11) on both sides of the equality we get

$$2(k_1^2(k_1^2 - j_1^2 + j_1'^2) - j_1^2 j_1'^2) = n^2(k_1^2 - j_1^2),$$

which implies

$$(k_1^2 - j_1^2)(n^2 - 2(k_1^2 + j_1'^2)) = 0.$$

Since n is odd, we necessarily have $n^2 \neq 2(k_1^2 + j_1'^2)$, which implies (jointly with (2.11))

$$k_1 = j_1, \quad k_1' = j_1'.$$

Equation (2.13) becomes

$$\frac{k_1^2}{4k_1^2 - n^2} \left(\frac{j_2^2}{(2\pi)^2 j_2^2 + n^2} - \frac{k_2^2}{(2\pi)^2 k_2^2 + n^2} \right) = \frac{k_1'^2}{4k_1'^2 - n^2} \left(\frac{j_2'^2}{(2\pi)^2 j_2'^2 + n^2} - \frac{k_2'^2}{(2\pi)^2 k_2'^2 + n^2} \right). \quad (2.17)$$

We are going to use several times the following technical result.

Lemma 2.8. *Let ξ be a transcendental number, and take $a, b, c, d, \gamma \in \mathbb{N}$ and $\mu \in \mathbb{Q} \setminus \{0\}$. If*

$$\frac{a}{a\xi + \gamma} - \frac{b}{b\xi + \gamma} = \mu \left(\frac{c}{c\xi + \gamma} - \frac{d}{d\xi + \gamma} \right) \quad (2.18)$$

then one of the properties holds true: (i) $(a, c) = (b, d)$, (ii) $\mu = 1$ and $(a, b) = (c, d)$, (iii) $\mu = -1$ and $(a, b) = (d, c)$.

Proof. The proof consists simply in noticing that (2.18) is equivalent to the equality

$$\frac{a}{aX + \gamma} - \frac{b}{bX + \gamma} = \mu \left(\frac{c}{cX + \gamma} - \frac{d}{dX + \gamma} \right)$$

between rational functions in the variable X and in comparing their poles. \square

Applying the lemma to the identity (2.17), we get that either $(j_2, j'_2) = (k_2, k'_2)$, and hence $(\mathbf{k}, \mathbf{k}') = (\mathbf{j}, \mathbf{j}')$ as desired, or $\{j_2, k_2\} = \{j'_2, k'_2\}$. In the latter case, moreover, (2.12) implies that $(j_2, k_2) = (j'_2, k'_2)$, which yields

$$\frac{k_1^2}{4k_1^2 - n^2} = \frac{k'_1{}^2}{4k'_1{}^2 - n^2},$$

since we are in case (ii) of Lemma 2.8. Since the map $x \mapsto x^2/(4x^2 - n^2)$ is injective on $[1, +\infty)$ then $k_1 = k'_1$, which contradicts (2.16).

Let now

$$k_1 = k'_1, \quad k_2 \neq k'_2. \quad (2.19)$$

Identity (2.11) implies that $j_1 = j'_1$ and equation (2.13) simplifies to

$$\frac{j_1^2}{4j_1^2 - n^2} \left(\frac{j_2^2}{(2\pi)^2 j_2^2 + n^2} - \frac{j_2'^2}{(2\pi)^2 j_2'^2 + n^2} \right) = \frac{k_1^2}{4k_1^2 - n^2} \left(\frac{k_2^2}{(2\pi)^2 k_2^2 + n^2} - \frac{k_2'^2}{(2\pi)^2 k_2'^2 + n^2} \right). \quad (2.20)$$

Let us apply again Lemma 2.8. Case (i) is ruled out by assumption (2.19). Hence, $\{j_2, j'_2\} = \{k_2, k'_2\}$ and it follows from (2.12), using the same argument as before, that $(j_2, j'_2) = (k_2, k'_2)$ and $j_1 = k_1$. We conclude also in this second case that $(\mathbf{k}, \mathbf{k}') = (\mathbf{j}, \mathbf{j}')$ and this concludes the proof of Proposition 2.7. \square

2.4.2 $\mathcal{P}_{2,BC}^0$ is nonempty

Let χ_n be defined as in the previous section (see equation (2.7)).

Proposition 2.9. *If n is even then $\chi_n \in \mathcal{P}_{2,BC}^0$.*

Proof. We use below the same parameterization on \mathbb{N}^2 of eigenpairs of the Laplace-Dirichlet operator as in Section 2.4.1. Notice that the notion of connectedness chain introduced in Definition 2.1 and (2.2) and (2.4) naturally extends to subsets of $(\mathbb{N}^2)^2$. Then, χ_n is in $\mathcal{P}_{2,BC}^0$ if and only if

$$\left\{ (\mathbf{j}, \mathbf{k}) \in (\mathbb{N}^2)^2 \mid \int_{\Omega} V_0^{\text{Id}, \chi_n}(x) \phi_{\mathbf{k}}(x) \phi_{\mathbf{j}}(x) dx \neq 0 \right\}$$

is a connectedness chain.

In order to prove that $S_0(\chi_n) = S(\Omega, \Gamma_D^g, \chi_n)$ is a connectedness chain, we are led to compute the quantities

$$\int_{\Omega} V_0^{\text{Id}, \chi_n}(x) \phi_{\mathbf{j}}(x) \phi_{\mathbf{k}}(x) dx = \frac{4}{L\pi} A_{n\mathbf{j}\mathbf{k}} B_{n\mathbf{j}\mathbf{k}}, \quad \mathbf{j}, \mathbf{k} \in (\mathbb{N}^2)^2,$$

with

$$A_{n\mathbf{j}\mathbf{k}} = \int_0^\pi \sin(nx_1) \sin(j_1 x_1) \sin(k_1 x_1) dx_1$$

and

$$B_{n\mathbf{j}\mathbf{k}} = \int_0^L \cosh(nx_2) \sin\left(\frac{j_2 \pi x_2}{L}\right) \sin\left(\frac{k_2 \pi x_2}{L}\right) dx_2.$$

A tedious but straightforward computation proves that

$$A_{n\mathbf{jk}} = \begin{cases} 0 & \text{if } j_1 + k_1 + n \text{ is even,} \\ \frac{-4j_1k_1n}{(j_1 + k_1 - n)(j_1 - k_1 + n)(-j_1 + k_1 + n)(j_1 + k_1 + n)} & \text{otherwise,} \end{cases}$$

whereas

$$B_{n\mathbf{jk}} = \frac{2(-1)^{j_2+k_2}L^2n\pi^2j_2k_2\sinh(nL)}{(\pi^2(j_2 - k_2)^2 + n^2)(\pi^2(j_2 + k_2)^2 + n^2)}.$$

One immediately sees that the coefficients $B_{n\mathbf{jk}}$ cannot vanish. As for the coefficients $A_{n\mathbf{jk}}$, if n is even then $A_{n\mathbf{jk}}$ vanishes if and only if j_1 and k_1 have the same parity. Then $S(\Omega, \Gamma_D^g, \chi_n) = \{(\mathbf{j}, \mathbf{k}) \mid j_1 + k_1 \text{ is odd}\}$ is a connectedness chain: indeed, given \mathbf{j} and \mathbf{k} in \mathbb{N}^2 , either $j_1 + k_1$ is odd, and then $(\mathbf{j}, \mathbf{k}) \in S(\Omega, \Gamma_D^g, \chi_n)$, or $j_1 + k_1$ is even and then $(\mathbf{j}, \mathbf{j}')$ and $(\mathbf{j}', \mathbf{k})$ are in $S(\Omega, \Gamma_D^g, \chi_n)$ with $\mathbf{j}' = (j_1 + 1, j_2)$. \square

Notice that, conversely, if n is odd then $A_{n\mathbf{jk}}$ vanishes if and only if $j_1 + k_1$ is odd. Hence, $S(\Omega, \Gamma_D^g, \chi_n)$ cannot couple \mathbf{j} and \mathbf{k} when $j_1 + k_1$ is odd. Therefore, $\chi_n \notin \mathcal{P}_{2, \text{BC}}^0$ for n odd.

2.5 Proof of Theorem 1.3

According to Proposition 2.3 and Corollary 2.6, we are left to prove that \mathcal{P}_1 is nonempty. Indeed, we already showed in the previous section that $\mathcal{P}_{2, \text{BC}}^0$ is nonempty, which implies that \mathcal{P}_2 , which contains $\{(\Omega, \Gamma_D^g, \chi) \mid \chi \in \mathcal{P}_{2, \text{BC}}^0\}$, is nonempty as well. We actually prove directly that \mathcal{P}_1 is residual, based on a general result proved in [43].

Lemma 2.10. *The set \mathcal{P}_1 is residual.*

Proof. Thanks to [43, Theorem 2.3], the lemma is proved if we show that for every $\ell \in \mathbb{N}$ and $q = (q_1, \dots, q_\ell) \in \mathbb{Q}^\ell \setminus \{0\}$ there exists $(Q(\Omega), Q(\Gamma_D^g), \chi) \in \mathcal{P}$ such that the first ℓ eigenvalues $\lambda_1, \dots, \lambda_\ell$ of the Dirichlet–Laplace operator on $Q(\Omega)$ are simple and $\sum_{j=1}^\ell q_j \lambda_j \neq 0$.

Fix $\ell \in \mathbb{N}$ and $q = (q_1, \dots, q_\ell) \in \mathbb{Q}^\ell \setminus \{0\}$. Let $\hat{L} > 0$ be such that $\pi^2 \ell^2 < \hat{L}^2$ and consider $\hat{\Omega} = (0, \pi) \times (0, \hat{L})$. The choice of \hat{L} is such that the ℓ smallest eigenvalues of $-\Delta$ on $\hat{\Omega}$ with Dirichlet boundary conditions are $\lambda_j = 1 + j^2 \pi^2 / \hat{L}^2$, which are simple and whose corresponding eigenfunctions are (up to normalization)

$$\phi_j(x_1, x_2) = \frac{2 \sin(x_1) \sin(j\pi x_2 / \hat{L})}{\sqrt{\pi \hat{L}}}.$$

Let X be a \mathcal{C}^1 vector field on \mathbb{R}^2 with compact support intersecting $\{0\} \times (0, \hat{L})$ but not any other side of $\hat{\Omega}$. For $t_0 > 0$ small enough and $t \in (-t_0, t_0)$, $I + tX$ is a diffeomorphism between $\hat{\Omega}$ and its image, which we will denote by $\hat{\Omega}_t$.

Denote by $(\lambda_j(t))_{j \in \mathbb{N}}$ the spectrum of the Laplace–Dirichlet operator on $\hat{\Omega}_t$. According to Rellich’s theorem (see [32, 45]), each function $\lambda_j(\cdot)$ can be chosen to be analytic on $(-t_0, t_0)$. Moreover, up to reducing t_0 , we can assume that $\lambda_1(t), \dots, \lambda_\ell(t)$ are simple for $t \in (-t_0, t_0)$.

It is well known that

$$\dot{\lambda}_j(0) = - \int_{\partial\hat{\Omega}} \left(\frac{\partial\phi_j}{\partial\nu} \right)^2 (X \cdot \nu)$$

for every j such that $\lambda_j(0)$ is simple (see, for instance, [27]). Notice that

$$\left(\frac{\partial\phi_j}{\partial\nu} \right)^2 = \frac{4 \sin^2(j\pi x_2/\hat{L})}{\pi \hat{L}}$$

on $\{0\} \times (0, \hat{L})$ for $j = 1, \dots, \ell$.

Henceforth, since $x_2 \mapsto \sin^2(j\pi x_2/\hat{L})$, $j = 1, \dots, \ell$, are linearly independent functions on $(0, \hat{L})$ (as it follows from the trigonometric formula $\sin^2\theta = 1 - \cos(2\theta)/2$ and by injectivity of Fourier series), then we can choose the vector field X in such a way that

$$\sum_{j=1}^{\ell} q_j \dot{\lambda}_j(0) \neq 0.$$

Hence, there exists $t \in (-t_0, t_0)$ such that $\sum_{j=1}^{\ell} q_j \lambda_j(t) \neq 0$ and the lemma is proved taking $Q = \text{Id} + tX$. \square

3 Generalizations

In this section we provide some generalizations of the results obtained in Theorems 1.2 and 1.3. In Section 3.1 we consider gates which do not cover the entire upper side of the rectangle Ω . In Section 3.2 we include some physically motivated nonlinear correction to the coupling term between the Poisson and the Schrödinger equation.

3.1 Partial gate with linear coupling

The model that we consider here is the following,

$$\left\{ \begin{array}{ll} i\partial_t \psi(t, x) = -\Delta \psi(t, x) + V(t, x)\psi(t, x), & (t, x) \in \mathbb{R}_+ \times \Omega, \\ -\Delta V(t, x) = 0, & (t, x) \in \mathbb{R}_+ \times \Omega, \\ \psi(t, x) = 0, & (t, x) \in \mathbb{R}_+ \times \partial\Omega, \\ V(t, x) = V_g(t)\chi(x), & (t, x) \in \mathbb{R}_+ \times \Gamma_D^g, \\ V(t, x) = 0, & (t, x) \in \mathbb{R}_+ \times \Gamma_D^s \cup \Gamma_D^d, \\ \frac{\partial V}{\partial\nu}(t, x) = 0, & (t, x) \in \mathbb{R}_+ \times \Gamma_N. \end{array} \right. \quad (3.1)$$

The set Ω still denotes the rectangle $(0, \pi) \times (0, L)$, $L > 0$. The gate Γ_D^g is now reduced to a compactly contained subinterval of $[0, \pi] \times \{L\}$, while $\Gamma_N = \Gamma_N^1 \cup \Gamma_N^2 \cup \Gamma_N^3$ is now the union of three connected components, as illustrated in Figure 2.

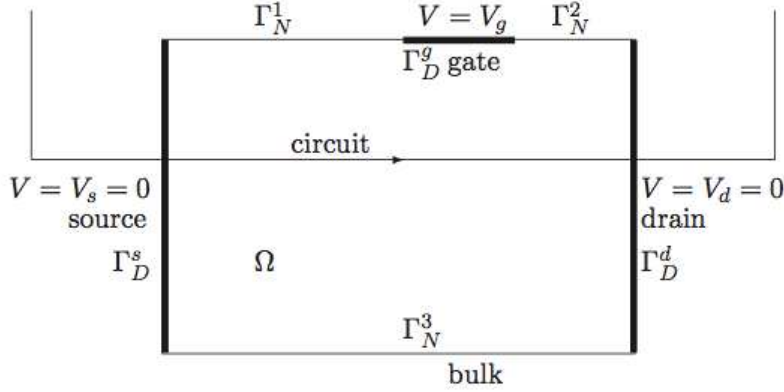


Figure 2: Representation of the transistor with partial gate

As in the previous sections, we can consider a deformation of Ω by introducing a transformation $Q \in \text{Diff}_0^1$, with $\chi \in \mathcal{C}^1(Q(\Gamma_D^g))$. Similarly to what is done in Section 1.2 and with a slight abuse of notations, we denote by \mathcal{P} the class of corresponding problems, identified with

$$\mathcal{P} = \{(Q(\Omega), Q(\Gamma_D^g), \chi) \mid (Q(\Omega), Q(\Gamma_D^g)) \in \Sigma \text{ and } \chi \in \mathcal{C}^1(Q(\Gamma_D^g))\}, \quad (3.2)$$

where

$$\Sigma = \{(Q(\Omega), Q(\Gamma_D^g)) \mid Q \in \text{Diff}_0^1\}.$$

We obtain the following result.

Theorem 3.1. *For a generic element of \mathcal{P} , the control problem (3.1) is approximately controllable.*

Proof. The proof consists in an adaptation of the one of Theorem 1.3. We denote by \mathcal{P}_1 and \mathcal{P}_2 the sets defined in analogy to what is done in Section 2.2. The same argument as in Proposition 2.3 allows us to prove the theorem by showing that \mathcal{P}_1 and \mathcal{P}_2 are residual.

Notice that the condition defining the set \mathcal{P}_1 actually depends only on $Q(\Omega)$, and not on $Q(\Gamma_D^g)$ and χ . Hence, as proved in Lemma 2.10, \mathcal{P}_1 is residual.

Let us focus on the set \mathcal{P}_2 . It is crucial for our argument to notice that the analyticity of $V_0^{Q,\chi}$ with respect to Q and χ still holds in the case of partial gates, as it can be seen by a straightforward adaptation of Proposition 2.5. As a consequence, as it was done in Corollary 2.6, it is sufficient to prove that the set \mathcal{P}_2 is nonempty. For that purpose we proceed by defining a suitable subclass of \mathcal{P} in which we are able to prove the density of \mathcal{P}_2 .

Indeed, consider $\tilde{L} > 0$ such that $\tilde{L}^2 \notin \pi^2\mathbb{Q}$ and define $\tilde{\Omega} = (0, \pi) \times (0, \tilde{L})$. Let us introduce the subclass $\tilde{\mathcal{P}}$ of \mathcal{P} defined by

$$\tilde{\mathcal{P}} = \{(Q(\Omega), Q(\Gamma_D^g), \chi) \in \mathcal{P} \mid Q(\Omega) = \tilde{\Omega}, Q(\Gamma_D^g) \subset [0, \pi] \times \{\tilde{L}\}\}.$$

Denote by $(\phi_{\mathbf{j}})_{\mathbf{j} \in \mathbb{N}^2}$ an L^2 -orthonormal basis for the Laplace–Dirichlet operator on $\tilde{\Omega}$.

Let $n \in \mathbb{N}$ be even, χ_n be defined as in (2.7) (see Proposition 2.9) and let

$$\bar{S} = S(Q(\Omega), Q(\Gamma_D^g), \chi_n).$$

The intersection of \mathcal{P}_2 with $\tilde{\mathcal{P}}$ contains in particular those elements $(Q(\Omega), Q(\Gamma_D^g), \chi) \in \tilde{\mathcal{P}}$ such that $\bar{S} \subset S(Q(\Omega), Q(\Gamma_D^g), \chi)$, i.e.,

$$\mathcal{P}_2 \cap \tilde{\mathcal{P}} \supset \bigcap_{(\mathbf{j}, \mathbf{k}) \in \bar{S}} \mathcal{O}_{\mathbf{j}\mathbf{k}}$$

where, for every $\mathbf{j}, \mathbf{k} \in \mathbb{N}^2$,

$$\mathcal{O}_{\mathbf{j}\mathbf{k}} = \{(Q(\Omega), Q(\Gamma_D^g), \chi) \in \tilde{\mathcal{P}} \mid \int_{\tilde{\Omega}} V_0^{Q, \chi} \phi_{\mathbf{j}} \phi_{\mathbf{k}} \neq 0\}.$$

Clearly, each $\mathcal{O}_{\mathbf{j}\mathbf{k}}$ is open in $\tilde{\mathcal{P}}$. The proof of the theorem is concluded by showing that $\mathcal{O}_{\mathbf{j}\mathbf{k}}$ is dense for every $(\mathbf{j}, \mathbf{k}) \in \bar{S}$. Actually, we just need to prove that for every $(\mathbf{j}, \mathbf{k}) \in \bar{S}$ there exists an element $(Q^{\mathbf{j}, \mathbf{k}}(\Omega), Q^{\mathbf{j}, \mathbf{k}}(\Gamma_{p,D}^g), \chi^{\mathbf{j}, \mathbf{k}})$ in $\mathcal{O}_{\mathbf{j}\mathbf{k}}$: indeed, any other element of $\tilde{\mathcal{P}}$ can be connected to $(Q^{\mathbf{j}, \mathbf{k}}(\Omega), Q^{\mathbf{j}, \mathbf{k}}(\Gamma_{p,D}^g), \chi^{\mathbf{j}, \mathbf{k}})$ by an analytic path within $\tilde{\mathcal{P}}$, along which $V_0^{Q, \chi}$ varies analytically (while $\phi_{\mathbf{j}}$ and $\phi_{\mathbf{k}}$ do not vary at all). In particular, almost every element of the path is in $\mathcal{O}_{\mathbf{j}\mathbf{k}}$, whence the density of $\mathcal{O}_{\mathbf{j}\mathbf{k}}$ in $\tilde{\mathcal{P}}$.

Let us introduce a sequence $(\Gamma_{p,D}^g)_{p \in \mathbb{N}}$ of segments included in $(0, \pi) \times \{\tilde{L}\}$ increasing for the inclusion and such that

$$\bigcup_{p=1}^{+\infty} \Gamma_{p,D}^g = (0, \pi) \times \{\tilde{L}\}.$$

For every $p \in \mathbb{N}$ let $Q_p \in \text{Diff}_0^1$ be such that $Q_p(\Omega) = \tilde{\Omega}$ and $Q_p(\Gamma_D^g) = \Gamma_{p,D}^g$, and

$$\eta_p = \chi_n \Big|_{\Gamma_{p,D}^g} \in \mathcal{C}^1(\Gamma_{p,D}^g).$$

The following continuity result holds true and concludes the proof of the theorem.

Lemma 3.2. *Define $V_0^p = V_0^{Q_p, \eta_p}$. The sequence $(V_0^p)_{p \in \mathbb{N}}$ converges strongly in $H^1(\tilde{\Omega})$ to V_0^{Id, χ_n} as $p \rightarrow +\infty$.*

By a slight notational abuse we denote by χ_n its extension on $\tilde{\Omega}$ satisfying $\chi_n(x_1, x_2) = \chi_n(x_1, \tilde{L})$ for every $(x_1, x_2) \in \tilde{\Omega}$. Let us introduce the lift $W_p = V_0^p - \chi_n$. Thus, W_p is the solution of the following partial differential equation

$$\begin{cases} -\Delta W_p(x) = n^2 \chi_n(x), & x \in (0, \pi) \times (0, \tilde{L}), \\ W_p(x) = 0, & x \in \Gamma_{p,D}^g \cup (\{0, \pi\} \times [0, \tilde{L}]), \\ \frac{\partial W_p}{\partial \nu}(x) = 0, & x \in ([0, \pi] \times \{0\}) \cup (([0, \pi] \times \{\tilde{L}\}) \setminus \Gamma_{p,D}^g), \end{cases} \quad (3.3)$$

whose variational formulation is written as follows: find W_p in

$$\mathcal{V}_p = \left\{ v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_{p,D}^g \cup (\{0, \pi\} \times [0, \tilde{L}]) \right\}$$

such that for every $v \in \mathcal{V}_p$, one has

$$\int_{\tilde{\Omega}} \nabla W_p(x) \cdot \nabla v(x) dx = n^2 \int_{\tilde{\Omega}} v(x) \chi_n(x) dx. \quad (3.4)$$

By definition, each V_0^p is harmonic and reaches its maximal and minimal values on the boundary of $\tilde{\Omega}$ at some points where the normal derivative of V_0^p does not vanish, as it follows from the Hopf maximum principle. Thus, $\|V_0^p\|_{L^\infty(\tilde{\Omega})} \leq \max_{x_1 \in [0, \pi]} |\chi_n(x)| \leq \cosh(n\tilde{L})$.

As a consequence, the sequences $(V_0^p)_{p \in \mathbb{N}}$ and $(W_p)_{p \in \mathbb{N}}$ are uniformly bounded (with respect to p) in $L^2(\tilde{\Omega})$. Taking now $v = W_p$ in (3.4) yields

$$\|\nabla W_p\|_{L^2(\tilde{\Omega})}^2 \leq n^2 \|\chi_n\|_{L^2(\tilde{\Omega})} \|W_p\|_{L^2(\tilde{\Omega})}.$$

The sequence $(W_p)_{p \in \mathbb{N}}$ is thus bounded in $H^1(\tilde{\Omega})$ and, from Rellich compactness embedding theorem, converges up to a subsequence weakly in $H^1(\tilde{\Omega})$ and strongly in $L^2(\tilde{\Omega})$ to some $W_\infty \in H^1(\tilde{\Omega})$. In the sequel, we will still denote by $(W_p)_{p \in \mathbb{N}}$ the considered subsequence. Taking tests functions v in $\mathcal{C}^\infty(\tilde{\Omega})$ with compact support in (3.4) yields immediately that W_∞ satisfies

$$-\Delta W_\infty = n^2 \chi_n$$

in distributional sense. By compactness of the trace operator, one has necessarily $W_\infty = 0$ on $\{0, \pi\} \times [0, \tilde{L}]$. Since the sequence $(\Gamma_{p,D}^g)_{p \in \mathbb{N}}$ is increasing for the inclusion and converges to $(0, \pi) \times \{\tilde{L}\}$, one sees that for any compact $K \subset (0, \pi) \times \{\tilde{L}\}$ there exists p_0 such that $W_p = 0$ on K for every $p \geq p_0$. Thus, one yields $W_\infty = 0$ on $(0, \pi) \times \{\tilde{L}\}$. Finally, since \mathcal{V}_p is increasing with respect to p , it is obvious that for every $p \in \mathbb{N}$ and $v \in \mathcal{V}_p$, W_∞ satisfies

$$\int_{\tilde{\Omega}} \nabla W_\infty(x) \cdot \nabla v(x) dx = n^2 \int_{\tilde{\Omega}} v(x) \chi_\infty(x) dx. \quad (3.5)$$

Introduce $\mathcal{V}_\infty = \bigcup_{p=0}^{+\infty} \mathcal{V}_p$, that is,

$$\mathcal{V}_\infty = \left\{ v \in H^1(\Omega) \mid v = 0 \text{ on } ((0, \pi) \times \{\tilde{L}\}) \cup (\{0, \pi\} \times [0, \tilde{L}]) \right\}.$$

It is clear that W_∞ satisfies (3.5) for every $v \in \mathcal{V}_\infty$. By taking $v = W_p$ in (3.4) and since $(W_p)_{p \in \mathbb{N}}$ converges strongly in $L^2(\tilde{\Omega})$ to W_∞ , it follows that $\|W_p\|_{H^1(\tilde{\Omega})}$ converges to $\|W_\infty\|_{H^1(\tilde{\Omega})}$ as $p \rightarrow +\infty$. Since $(W_p)_{p \in \mathbb{N}}$ also converges weakly in $H^1(\tilde{\Omega})$ to W_∞ , we deduce that this convergence is in fact strong in $H^1(\tilde{\Omega})$, whence the result. \square

3.2 Nonlinear coupling

In this section, we show how the approximate controllability results proved in the previous sections can be applied to obtain some suitable controllability property for a nonlinear system. We now take into account selfconsistent electrostatic interactions between electrons in the Poisson equation. For simplicity, we only consider the case where the gate covers the entire upper side of the domain Ω .

We consider here the following Schrödinger–Poisson system,

$$\left\{ \begin{array}{ll} i\partial_t \psi(t, x) = -\Delta \psi(t, x) + V(t, x)\psi(t, x), & (t, x) \in \mathbb{R}_+ \times \Omega, \\ -\Delta V(t, x) = \alpha |\psi(t, x)|^2, & (t, x) \in \mathbb{R}_+ \times \Omega, \\ \psi(t, x) = 0, & (t, x) \in \mathbb{R}_+ \times \partial\Omega, \\ V(t, x) = V_g(t)\chi(x), & (t, x) \in \mathbb{R}_+ \times \Gamma_D^g, \\ V(t, x) = 0, & (t, x) \in \mathbb{R}_+ \times \Gamma_D^s \cup \Gamma_D^d, \\ \frac{\partial V}{\partial \nu}(t, x) = 0, & (t, x) \in \mathbb{R}_+ \times \Gamma_N. \end{array} \right. \quad (3.6)$$

Here, $\alpha > 0$ denotes a dimensionless parameter that quantifies the strength of nonlinear effects; $1/\sqrt{\alpha}$ is the so-called *scaled Debye length*. The domain Ω is the rectangle $(0, \pi) \times (0, L)$ in the configuration of Figure 1: the gate is the entire segment $\Gamma_D^g = [0, \pi] \times \{L\}$ and the Neumann boundary is $\Gamma_N = [0, \pi] \times \{0\}$.

In order to exploit elliptic regularity properties, we consider here smoother perturbation parameters than in previous sections, taking the diffeomorphism Q in Diff_0^2 (the class of \mathcal{C}^2 orientation-preserving diffeomorphism of \mathbb{R}^2) and $\chi \in \mathcal{C}_0^1(Q(\Gamma_D^g))$.

It is convenient to split the potential into the sum of the control potential and the nonlinear potential as $V(t, x) = V_g(t)V_0(x) + W_\psi(t, x)$. The resulting equation in the deformed domain can be written as

$$i\partial_t \psi = -\Delta \psi + V_g(t)V_0\psi + W_\psi\psi, \quad \text{in } \mathbb{R}_+ \times Q(\Omega), \quad (3.7)$$

where V_0 and W_ψ are the solutions of

$$\left\{ \begin{array}{ll} -\Delta V_0(x) = 0, & x \in Q(\Omega), \\ V_0(x) = \chi(x), & x \in Q(\Gamma_D^g), \\ V_0(x) = 0, & x \in Q(\Gamma_D^s \cup \Gamma_D^d), \\ \frac{\partial V_0}{\partial \nu}(x) = 0, & x \in Q(\Gamma_N), \end{array} \right. \quad (3.8)$$

and

$$\left\{ \begin{array}{ll} -\Delta W_\psi(t, x) = \alpha |\psi(t, x)|^2, & (t, x) \in \mathbb{R}_+ \times Q(\Omega), \\ W_\psi(t, x) = 0, & (t, x) \in \mathbb{R}_+ \times Q(\Gamma_D^g \cup \Gamma_D^s \cup \Gamma_D^d), \\ \frac{\partial W_\psi}{\partial \nu}(t, x) = 0, & (t, x) \in \mathbb{R}_+ \times Q(\Gamma_N). \end{array} \right. \quad (3.9)$$

Before stating our approximate controllability result for (3.7), we address the question of well-posedness of this Cauchy problem. Two kinds of results are available for

Schrödinger–Poisson systems, see [18]. In the whole-space case $\Omega = \mathbb{R}^2$, Strichartz estimates enable to benefit from the dispersive and smoothing properties of the Schrödinger group and construct a unique global L^2 solution to the problem [24, 25, 17]. In a general domain Ω , for more regular initial data in H^1 or H^2 , the analysis is simpler and the proof of global well-posedness can rely on energy estimates, see [2, 16, 29, 34].

However, none of these results apply to our situation, which requires a specific study. Indeed, dealing with a problem set on general bounded domains $Q(\Omega)$, our analysis cannot rely on Strichartz estimate and we have to assume that the Cauchy data are more regular than L^2 , for instance that they belong to the energy space H^1 . In this case, the proof of local in time existence and uniqueness of a solution to (3.7) is not a difficult task and the main issue is the question of *global* existence. As we said above, the proof of global existence usually relies on an energy estimate for (3.7), (3.8), (3.9). When the applied potential $V_g(t)$ is differentiable, this estimate can be obtained by multiplying (3.7) by $\partial_t \bar{\psi}$ and integrating on $[0, t] \times Q(\Omega)$, and reads

$$\begin{aligned} \|\nabla \psi(t)\|_{L^2(Q(\Omega))}^2 + \frac{1}{2\alpha} \|\nabla W_\psi(t)\|^2 + V_g(t) \int_{Q(\Omega)} V_0(x) |\psi(t, x)|^2 dx \\ = \|\nabla \psi_0\|_{L^2(Q(\Omega))}^2 + \frac{1}{2\alpha} \|\nabla W_{\psi_0}\|^2 + V_g(0) \int_{Q(\Omega)} V_0(x) |\psi_0(x)|^2 dx \\ + \int_0^t \int_{Q(\Omega)} \partial_t V_g(s) V_0(x) |\psi(s, x)|^2 ds dx. \end{aligned}$$

For completeness, we consider in the following nonsmooth control functions $V_g \in L^\infty([0, T], [0, \delta])$, where $\delta > 0$ is given: for instance, V_g can be piecewise constant. In the general case, we follow another path to prove that the energy of the system –say the H^1 norm of ψ – remains bounded on any $[0, T]$, independently of the derivative of the control. We state this result in the following proposition, whose proof is based on a Brézis–Gallouet type argument [15].

Proposition 3.3. *Let $T > 0$, let $Q \in \text{Diff}_0^2$, let $\chi \in \mathcal{C}_0^1(Q(\Gamma_D^g))$, and let $V_g \in L^\infty([0, T], [0, \delta])$. Then, for every $\psi_0 \in H_0^1(Q(\Omega))$, the system (3.7), (3.8), (3.9) admits a unique mild solution $\psi \in \mathcal{C}^0([0, T], H_0^1(Q(\Omega)))$ and there exists $c > 0$ such that, for all $t \in [0, T]$,*

$$\|\psi(t, \cdot)\|_{H^1} \leq \exp(ce^{ct}). \quad (3.10)$$

The constant c only depends on δ , α_0 , Q , $\|\psi_0\|_{H^1}$ and $\|\chi\|_{\mathcal{C}^1}$.

Proof. Let us first prove the local well-posedness of the Cauchy problem in $H_0^1(Q(\Omega))$. For all $T_0 > 0$, we set $X_{T_0} = \mathcal{C}^0([0, T_0], H_0^1(Q(\Omega)))$ with the norm

$$\|u\|_{X_{T_0}} = \max_{t \in [0, T_0]} \|\nabla u(t)\|_{L^2}.$$

Denoting by $(e^{i\tau\Delta})_{\tau \in \mathbb{R}}$ the group of unitary transformations generated by the operator $i\Delta$ with Dirichlet boundary conditions, a mild solution $\psi \in X_{T_0}$ of (3.7) satisfies

$$\psi(t, \cdot) = e^{-it\Delta} \psi_0(\cdot) + \int_0^t e^{-i(t-s)\Delta} (V_g(s) V_0(\cdot) + W_\psi(s, \cdot)) \psi(s, \cdot) ds, \quad (3.11)$$

where V_0 and W_ψ are defined by (3.8) and (3.9), and can be characterized as a fixed-point of the mapping $S : X_{T_0} \rightarrow X_{T_0}$ given by

$$S(\psi) = e^{-it\Delta}\psi_0(\cdot) + \int_0^t e^{-i(t-s)\Delta}(V_g(s)V_0(\cdot) + W_\psi(s, \cdot))\psi(s, \cdot)ds.$$

Let $R > \|\nabla\psi_0\|_{L^2}$ be fixed and define

$$\mathcal{B}_R = \{u \in X_{T_0} : \|u\|_{X_{T_0}} \leq R\}.$$

We will prove that, for T_0 small enough, S is a contraction mapping on \mathcal{B}_R .

By elliptic regularity, since the function χ belongs to \mathcal{C}^1 and vanishes at the boundary of the grid, the fixed potential V_0 which solves (3.8) belongs (at least) to $H^{3/2}(Q(\Omega))$: see [21], chapter 8, concerning the mixed problem for the Laplacian in curved domains with corners, and [31] for the treatment of the Dirichlet-Dirichlet corners.

Denoting in the following by C any positive constant depending only on the domain $Q(\Omega)$, the Sobolev embeddings $H^{3/2} \hookrightarrow W^{1,4}$, $H^1 \hookrightarrow L^4$, $H^{3/2} \hookrightarrow L^\infty$ and the Poincaré inequality yield

$$\begin{aligned} \|\nabla(V_0\psi)\|_{L^2} &\leq \|(\nabla V_0)\psi\|_{L^2} + \|V_0\nabla\psi\|_{L^2} \leq \|\nabla V_0\|_{L^4}\|\psi\|_{L^4} + \|V_0\|_{L^\infty}\|\nabla\psi\|_{L^2} \\ &\leq C\|V_0\|_{H^{3/2}}\|\nabla\psi\|_{L^2}. \end{aligned} \quad (3.12)$$

By elliptic regularity and Sobolev embeddings, we have for all $\psi, \tilde{\psi} \in H_0^1(\Omega)$

$$\begin{aligned} \|W_\psi - W_{\tilde{\psi}}\|_{H^2} &\leq C\alpha\|\psi\|^2 - \|\tilde{\psi}\|^2\|_{L^2} \leq C\alpha(\|\psi\|_{L^4} + \|\tilde{\psi}\|_{L^4})\|\psi - \tilde{\psi}\|_{L^4} \\ &\leq C\alpha(\|\nabla\psi\|_{L^2} + \|\nabla\tilde{\psi}\|_{L^2})\|\nabla(\psi - \tilde{\psi})\|_{L^2}, \end{aligned} \quad (3.13)$$

so, proceeding as for (3.12), we get

$$\|\nabla(W_\psi\psi - W_{\tilde{\psi}}\tilde{\psi})\|_{L^2} \leq C\alpha(\|\nabla\psi\|_{L^2}^2 + \|\nabla\tilde{\psi}\|_{L^2}^2)\|\nabla(\psi - \tilde{\psi})\|_{L^2}.$$

Finally, using that $e^{-it\Delta}$ is unitary on $H_0^1(Q(\Omega))$, we obtain, for all $\psi, \tilde{\psi} \in \mathcal{B}_R$

$$\begin{aligned} \|S(\psi)\|_{X_{T_0}} &\leq \|\nabla\psi_0\|_{L^2} + \int_0^{T_0} |V_g(s)| (\|\nabla(V_0\psi)(s)\|_{L^2} + \|\nabla(W_\psi\psi)(s)\|_{L^2}) ds \\ &\leq \|\nabla\psi_0\|_{L^2} + CT_0 (\|V_g\|_{L^\infty}\|V_0\|_{H^{3/2}}R + \alpha R^3) \end{aligned}$$

and

$$\begin{aligned} \|S(\psi) - S(\tilde{\psi})\|_{X_{T_0}} &\leq \int_0^{T_0} |V_g(s)| (\|\nabla(V_0(\psi - \tilde{\psi}))(s)\|_{L^2} + \|\nabla(W_\psi\psi - W_{\tilde{\psi}}\tilde{\psi})(s)\|_{L^2}) ds \\ &\leq CT_0 (\|V_g\|_{L^\infty}\|V_0\|_{H^{3/2}} + 2\alpha R^2) \|\psi - \tilde{\psi}\|_{X_{T_0}}. \end{aligned}$$

Hence, it is clear that, since $\|\nabla\psi_0\|_{L^2} < R$, choosing T_0 small enough ensures $S(\psi) \in \mathcal{B}_R$ and $\|S(\psi) - S(\tilde{\psi})\|_{X_{T_0}} < q\|\psi - \tilde{\psi}\|_{X_{T_0}}$ with $q < 1$. Then, the Banach fixed-point theorem implies the existence of a unique mild solution to (3.7) on the time interval $[0, T_0]$.

Furthermore, if the *a priori* estimate (3.10) is proved, then by a standard continuation argument, the existence interval can be taken equal to $[0, T]$, which means that the solution is in fact global in time.

Let us now prove the crucial estimate (3.10). We first recall that the L^2 norm of ψ is an invariant of (3.7): for all $t \geq 0$, one has $\|\psi(t)\|_{L^2} = \|\psi_0\|_{L^2}$. To estimate the H_0^1 norm of ψ , we come back to (3.11) which yields

$$\begin{aligned} \|\nabla\psi(t)\|_{L^2} &\leq \|\nabla\psi_0\|_{L^2} + \|V_g\|_{L^\infty} \int_0^t \|\nabla(V_0\psi)(s)\|_{L^2} ds + \int_0^t \|\nabla(W_\psi\psi)(s)\|_{L^2} ds \\ &\leq \|\nabla\psi_0\|_{L^2} + C\|V_g\|_{L^\infty}\|V_0\|_{H^{3/2}} \int_0^t \|\nabla\psi(s)\|_{L^2} ds + \int_0^t \|\nabla(W_\psi\psi)(s)\|_{L^2} ds, \end{aligned} \quad (3.14)$$

where we used (3.12). We thus need to estimate the product

$$\|\nabla(W_\psi\psi)\|_{L^2} \leq \|(\nabla W_\psi)\psi\|_{L^2} + \|W_\psi\nabla\psi\|_{L^2} \leq \|\nabla W_\psi\|_{L^4}\|\psi\|_{L^4} + \|W_\psi\|_{L^\infty}\|\nabla\psi\|_{L^2}. \quad (3.15)$$

For the first term, we use elliptic regularity and Sobolev embedding,

$$\|\nabla W_\psi\|_{L^4} \leq C\|W_\psi\|_{W^{2,4/3}} \leq C\alpha\|\psi\|_{L^{4/3}}^2 = C\alpha\|\psi\|_{L^{8/3}}^2.$$

Next, we recall the following two Gagliardo–Nirenberg inequalities: for all $\psi \in H_0^1(Q(\Omega))$, one has

$$\|\psi\|_{L^{8/3}}^2 \leq \|\nabla\psi\|_{L^2}^{1/2}\|\psi\|_{L^2}^{3/2}, \quad \|\psi\|_{L^4} \leq \|\nabla\psi\|_{L^2}^{1/2}\|\psi\|_{L^2}^{1/2}.$$

Hence, the first term in the right hand side of (3.15) can be bounded linearly in $\|\nabla\psi\|_{L^2}$ as

$$\|\nabla W_\psi\|_{L^4}\|\psi\|_{L^4} \leq C\alpha\|\psi\|_{L^2}^2\|\nabla\psi\|_{L^2} = C\alpha\|\psi_0\|_{L^2}^2\|\nabla\psi\|_{L^2}. \quad (3.16)$$

The main source of concern is the second term in the right hand side of (3.15). Indeed, $\|W_\psi\|_{L^\infty}$ cannot be bounded by a quantity which only depends on the L^2 norm of ψ (an L^1 right-hand side in the elliptic equation (3.9) does not produce an L^∞ potential), so this term will necessarily lead to a super-linear estimate in $\|\nabla\psi\|_{L^2}$.

A key inequality in the proof will be the following one, proved by Brézis and Gallouet in [15]. There exists a constant $C > 0$ such that, for all $u \in H^2(\Omega)$, one has

$$\|u\|_{L^\infty} \leq C(1 + \|u\|_{H^1}\sqrt{\log(1 + \|u\|_{H^2})}). \quad (3.17)$$

Let us multiply the first equation of (3.9) by W_ψ and integrate on $Q(\Omega)$. After an integration by parts, it comes

$$\|\nabla W_\psi\|_{L^2}^2 = \alpha \int_{Q(\Omega)} W_\psi |\psi|^2 dx \leq \alpha \|W_\psi\|_{L^\infty} \|\psi_0\|_{L^2}^2. \quad (3.18)$$

Therefore, from the Poincaré inequality, from (3.17) and (3.18), we deduce

$$\|W_\psi\|_{L^\infty}^2 \leq C(1 + \alpha\|W_\psi\|_{L^\infty}\|\psi_0\|_{L^2}^2 \log(1 + \|W_\psi\|_{H^2})),$$

from which we get

$$\|W_\psi\|_{L^\infty} \leq C(1 + \alpha\|\psi_0\|_{L^2}^2 \log(1 + \|W_\psi\|_{H^2})).$$

Next, using (3.13) with $\tilde{\psi} = 0$, we obtain

$$\|W_\psi\|_{L^\infty} \leq C(1 + \alpha\|\psi_0\|_{L^2}^2 \log(1 + \sqrt{\alpha}\|\nabla\psi\|_{L^2})). \quad (3.19)$$

Finally, gathering (3.14), (3.15), (3.16) and (3.19), one gets

$$\begin{aligned} \|\nabla\psi(t)\|_{L^2} &\leq \|\nabla\psi_0\|_{L^2} + C\|V_g\|_{L^\infty}\|V_0\|_{H^{3/2}} \int_0^t \|\nabla\psi(s)\|_{L^2} ds \\ &\quad + C(1 + \alpha\|\psi_0\|_{L^2}^2) \int_0^t (1 + \log(1 + \sqrt{\alpha}\|\nabla\psi(s)\|_{L^2})) \|\nabla\psi(s)\|_{L^2} ds \\ &\leq \|\nabla\psi_0\|_{L^2} + C\delta\|\chi\|_{C^1} \int_0^t \|\nabla\psi(s)\|_{L^2} ds \\ &\quad + C(1 + \alpha\|\psi_0\|_{L^2}^2) \int_0^t (1 + \log(1 + \sqrt{\alpha_0}\|\nabla\psi(s)\|_{L^2})) \|\nabla\psi(s)\|_{L^2} ds \end{aligned}$$

where we used $\|V_g\|_{L^\infty} \leq \delta$ and $\|V_0\|_{H^{3/2}} \leq C\|\chi\|_{C^1}$. A logarithmic Gronwall lemma (see [15]) yields the *a priori* estimate (3.10). The proof of the proposition is complete. \square

As application of this proposition, one deduces the following approximate controllability result for the nonlinear problem (3.7).

Theorem 3.4. *For a generic triple $(Q(\Omega), Q(\Gamma_g^D), \chi)$ in \mathcal{P} , defined as in (1.5), for every $\psi_0 \in H_0^1(Q(\Omega), \mathbb{C})$, $\psi_1 \in L^2(Q(\Omega), \mathbb{C})$, with $\|\psi_0\|_{L^2} = \|\psi_1\|_{L^2} = 1$, for every tolerance $\varepsilon > 0$, there exist a positive time T , a control $V_g \in L^\infty([0, T], [0, \delta])$, and $\alpha_0 > 0$ such that, if $0 < \alpha \leq \alpha_0$, then the solution of (3.7) satisfies $\|\psi(T) - \psi_1\|_{L^2(Q(\Omega))} < \varepsilon$.*

Proof. Recall that, by Theorem 1.3, for a generic triple $(Q(\Omega), Q(\Gamma_g^D), \chi)$ in \mathcal{P} , the linear system (1.3) is approximately controllable. Fix then Q and χ such that (1.3) is approximately controllable. Fix $\psi_0 \in H_0^1(Q(\Omega), \mathbb{C})$, $\psi_1 \in L^2(Q(\Omega), \mathbb{C})$, with $\|\psi_0\|_{L^2} = \|\psi_1\|_{L^2} = 1$ and $\varepsilon > 0$. Then there exist $T > 0$ and $V_g \in L^\infty([0, T], [0, \delta])$ such that the solution ψ_{lin} of the linear equation (1.3) with initial condition ψ_0 corresponding to V_g satisfies $\|\psi_{\text{lin}}(T) - \psi_1\|_{L^2} < \varepsilon/2$.

Then, the solution $\psi(t, x)$ of the nonlinear equation (3.7) with initial condition ψ_0 corresponding to the control V_g reads

$$\begin{aligned} \psi(t, \cdot) &= e^{-it\Delta}\psi_0(\cdot) + \int_0^t e^{-i(t-s)\Delta}(V_g(s)V_0(\cdot) + W_\psi(s, \cdot))\psi(s, \cdot)ds \\ &= \psi_{\text{lin}}(t, \cdot) + \int_0^t e^{-i(t-s)\Delta}W_\psi(s, \cdot)\psi(s, \cdot)ds. \end{aligned} \quad (3.20)$$

The L^∞ norm of W_ψ can be estimated by using elliptic regularity for (3.9), a Sobolev embedding and the bound (3.10) given in Proposition 3.3: for all $t \leq T$,

$$\begin{aligned} \|W_\psi(t)\|_{L^\infty} &\leq C\|W_\psi(t)\|_{H^2} \leq \alpha C\|\psi(t)\|_{L^2}^2 = \alpha C\|\psi(t)\|_{L^4}^2 \\ &\leq \alpha C\|\psi(t)\|_{H^1}^2 \leq \alpha C \exp(ce^{cT}). \end{aligned}$$

Fixing an upper bound $\alpha_{\max} > 0$ for α , the constant $c > 0$ can be chosen independent of α and depending only on Q , χ , ψ_0 and δ , which are all fixed. Hence, inserting this estimate in (3.20) yields

$$\begin{aligned} \|\psi(T) - \psi_{\text{lin}}(T)\|_{L^2} &\leq \int_0^T \|W_\psi(s, \cdot)\psi(s, \cdot)\|_{L^2} ds \leq \int_0^T \|W_\psi(s, \cdot)\|_{L^\infty} \|\psi(s, \cdot)\|_{L^2} ds \\ &\leq \alpha CT \exp(ce^{cT}), \end{aligned}$$

where we used $\|\psi\|_{L^2} = 1$ and that $e^{-i\tau\Delta}$ preserves the L^2 -norm. Then it suffices to take

$$\alpha_0 = \min \left(\alpha_{\max}, \frac{\varepsilon}{2CT \exp(ce^{cT})} \right)$$

and the theorem is proved. \square

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